TRINITY COLLEGE

Semester One Examination, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1
Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ____________________

Your name ____________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
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Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	57	35
Section Two: Calculator-assumed	13	13	100	106	65
			Total	163	100

Instructions to candidates

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2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
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5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

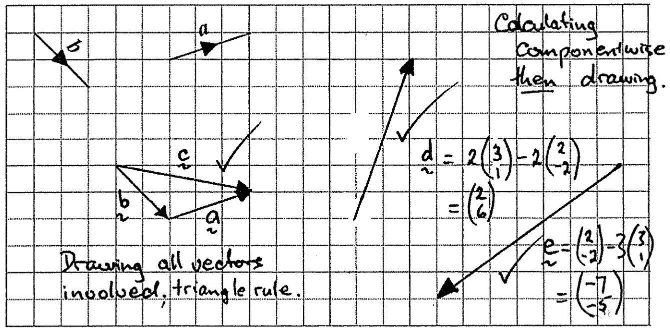
Section One: Calculator-free **(57 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(9 marks)**

(a) Two vectors, **a** and **b**, are shown on the grid below.



Draw and label the vectors **c**, **d** and **e** on the grid, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = 2\mathbf{a} - 2\mathbf{b}$ and $\mathbf{e} = \mathbf{b} - 3\mathbf{a}$. (3 marks)

(b) Determine a unit vector perpendicular to the vector $8\mathbf{i} - 6\mathbf{j}$. $\hat{u} = \frac{1}{|u|}u$ (3 marks)

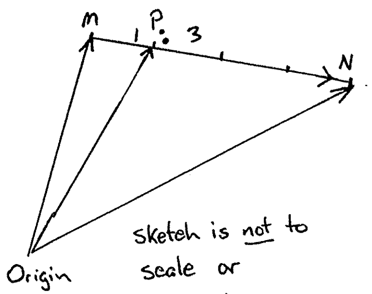
N.B. $(8i - 6j) \bullet p = 0$

$\hat{p} = \frac{1}{10}(6, 8)$ ✓✓ OR $\hat{p} = \frac{1}{10}(-6, -8)$

$= 0.6i + 0.8j$ ✓ accept either. $= -0.6i - 0.8j$

$= \frac{3}{5}i + \frac{4}{5}j$ $= -\frac{3}{5}i - \frac{4}{5}j$

(c) The point P divides the line segment from M(-3, 3) to N(13, -9) in the ratio 1:3. Determine the position vector of point P. (3 marks)



Sketch is not to scale or orientation, rather to provide a plan (strategy)

$p = \vec{OM} + \frac{1}{4}\vec{MN}$

$= \vec{OM} + \frac{1}{4}(\vec{ON} - \vec{OM})$ ✓

$= \binom{-3}{3} + \frac{1}{4}\left(\binom{13}{-9} - \binom{-3}{3}\right)$ ✓

$= \binom{-3}{3} + \frac{1}{4}\binom{16}{-12}$

$= \binom{1}{0}$

$\therefore P(1, 0)$ ✓

Question 2 **(6 marks)**

The statement 'if two rectangles are congruent then they have the same area' is true.

(a) Write the inverse of the statement and explain if the inverse is also true. (2 marks)

"If two rectangles are not congruent then they do not have the same area." ✓

False: e.g. 2×6 and 3×4 (Counter Example) are not congruent but have the same area 12. ✓

(b) Write the contrapositive of the statement and explain if the contrapositive is also true. (2 marks)

"If two rectangles do not have the same area then they are not congruent." ✓

True: all contrapositive statements are true. ✓

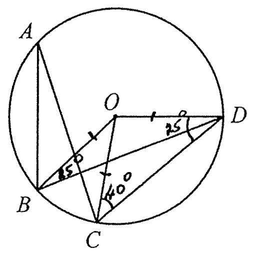
(c) Write the converse of the statement and explain if the converse is also true. (2 marks)

"If two rectangles have the same area then they are congruent." ✓

False: eg. 2×6 and 3×4 have the same area but they are not congruent. ✓

Question 3 **(7 marks)**

(a) In the diagram below, $\angle OBD = 25°$ and $\angle OCD = 40°$.



thoughts:
Equal radii
Isosceles triangles
Angle @ centre = Twice @ circum.
Angle @ circum same arc.

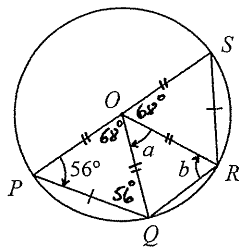
Determine the sizes of

(i) $\angle BDC$. $= 40° - 25°$ $= \underline{\underline{15°}}$ (1 mark)

(ii) $\angle BOC$. $= 2 \times 15°$ $= \underline{\underline{30°}}$ (1 mark)

(iii) $\angle CAB$. $= \underline{\underline{15°}}$ (1 mark)

(b) Determine, with reasons, the size of the angles marked a and b in the diagram below. (4 marks)



$\angle OQP = 56°$ Isosceles $\triangle$

$\angle POQ = 68°$ Angle sum of $\triangle$

$\angle SOR = 68°$ since SR ≅ PQ given

$\therefore a = 180° - 2(68°)$ straight angle

$= \underline{\underline{44°}}$

$b = \frac{180-44}{2}$ Isosceles $\triangle$

$= \underline{\underline{68°}}$

Question 4 **(9 marks)**

(a) Simplify $\frac{28! \times 7!}{10! \times 26!}$. (2 marks)

$= \frac{\overset{7}{\cancel{28}} \times \overset{3}{\cancel{27}} \times \cancel{26!} \times \cancel{7!}}{10 \times 9 \times \underset{2}{\cancel{8}} \times \cancel{7!} \times \cancel{26!}}$

$= \frac{7 \times 3}{10 \times 2}$

$= \underline{\underline{\frac{21}{20}}}$

(b) Prove that ${}^nP_r = n \times {}^{n-1}P_{r-1}$. (3 marks)

Proof

Take LHS $= {}^nP_r$

$= \frac{n!}{(n-r)!}$

$= \frac{n\,(n-1)!}{(n-r+1-1)!}$

$= n \times \frac{(n-1)!}{((n-1)-(r-1))!}$

$= n \times {}^{n-1}P_{r-1}$

$=$ RHS.

QED.

Proof

Take RHS $= n \times {}^{n-1}P_{r-1}$

$= n \times \frac{(n-1)!}{(n-1-r+1)!}$

$= \frac{n!}{(n-r)!}$

$= {}^nP_r$

$=$ LHS.

QED.

(c) If ${}^9P_3 = 504$ and ${}^{10}P_6 = 151200$, determine

(i) ${}^9P_5$. (2 marks)

From (b)

$${}^{10}P_6 = 10 \times {}^9P_5$$

$$\Rightarrow {}^9P_5 = \frac{{}^{10}P_6}{10} = \frac{151200}{10} = \underline{\underline{15120}}$$

(ii) ${}^{11}P_5$. (2 marks)

From (b)

$${}^{11}P_5 = 11 \times {}^{11}P_4$$

$$= 11 \times 10 \times {}^9P_3$$ and (b) again!

$$= 11 \times 10 \times 504$$

$$= \underline{\underline{55440}}$$

Question 5 **(10 marks)**

The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\mathbf{a} = (5,12)$ and $\mathbf{b} = (2,-1)$.

(a) Determine

(i) $\mathbf{a} - 3\mathbf{b}$. (2 marks)

$$= \begin{pmatrix} 5 \\ 12 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -1 \\ 15 \end{pmatrix}}}$$

(ii) $|\mathbf{a}| \times |\mathbf{b}|$. (2 marks)

$$= 13 \times \sqrt{5}$$

$$= \underline{\underline{13\sqrt{5}}}$$

F.Y.I.
(5,12,13) is a
Pythagorean Triple!

(iii) the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. (2 marks)

$$\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|} \cdot \hat{\underset{\sim}{b}} = \frac{5 \times 2 + 12(-1)}{\sqrt{5}} \times \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= -\frac{2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= (-0.8,\ 0.4)$$ i.e. $\left(-\frac{4}{5}, \frac{2}{5}\right)$

(b) Determine the vectors $\mathbf{c}$ and $\mathbf{d}$ if $2\mathbf{c} - 3\mathbf{d} = \mathbf{a}$ and $\mathbf{c} - 2\mathbf{d} = 2\mathbf{b}$. (4 marks)

$2\underset{\sim}{c} - 3\underset{\sim}{d} = \underset{\sim}{a}$ and $\underset{\sim}{c} - 2\underset{\sim}{d} = 2\underset{\sim}{b}$

$\Rightarrow \underset{\sim}{c} = 2\underset{\sim}{b} + 2\underset{\sim}{d}$

$\Rightarrow 2(2\underset{\sim}{b} + 2\underset{\sim}{d}) - 3\underset{\sim}{d} = \underset{\sim}{a}$

$\Rightarrow 4\underset{\sim}{b} + 4\underset{\sim}{d} - 3\underset{\sim}{d} = \underset{\sim}{a}$

$\Rightarrow \underset{\sim}{d} = \underset{\sim}{a} - 4\underset{\sim}{b}$

$= \begin{pmatrix} 5 \\ 12 \end{pmatrix} - 4\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

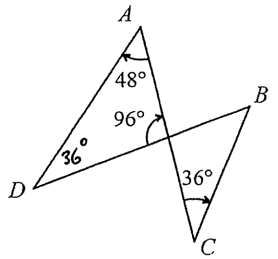
$= \underline{\underline{(-3, 16)}}$

$= 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -3 \\ 16 \end{pmatrix}$

$= \underline{\underline{(-2, 30)}}$

Question 6 **(7 marks)**

(a) Prove that it is possible to draw a circle through the points A, B, C and D shown below. (3 marks)

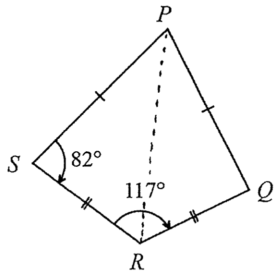


$\angle D = 180^\circ - (48^\circ + 96^\circ)$ Sum of angles in $\triangle$.
$= 36^\circ$
$= \angle C$

Now if points $A + B$ lie on a circle then D and C lie on the same circle (angles @ circumference standing on same arc are equal)

QED.

(b) Prove by contradiction that it is impossible to draw a circle through the vertices of the quadrilateral shown below. (4 marks)



Assume PQRS is a cyclic quadrilateral.
i.e. $\angle S + \angle Q = 180^\circ$

$\triangle PRS \cong \triangle PRQ$ (SSS)
$\Rightarrow \angle Q = \angle S = 82^\circ$ corresponding angles of congruent $\triangle$
and so $\angle S + \angle Q$
$= 82^\circ + 82^\circ$
$= 164^\circ$
$\neq 180^\circ$ contradiction!

$\therefore$ Not possible to be a cyclic quad.

QED.

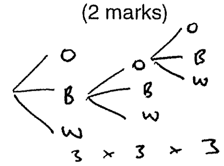
Question 7 **(9 marks)**

(a) A bag contains 17 identical cubes except for their colour, with four coloured orange, six coloured blue and seven coloured white.

(i) How many different arrangements of coloured cubes are possible when three cubes are drawn from the bag and placed in a line? (2 marks)

1st 2nd 3rd
$3 \times 3 \times 3$
$= \underline{\underline{27}}$ arrangements

*(O, O, O)
*(B, B, B) three colours.
etc
*(W, W, W)



(ii) How many different combinations of coloured cubes are possible when three cubes are drawn from the bag? (2 marks)

All the same OR one different (i.e. two same) OR all different
$(3 \times 1) + (3 \times 2) + 1$
* $3 + 6 + 1$ (one of each)
$= \underline{\underline{10}}$ combinations

(iii) Determine the least number of cubes that should be removed from the bag to ensure that the resulting selection contains at least three cubes of one colour. Justify your answer. (2 marks)

6 cubes could produce two of each colour (3×2)

$\underline{\underline{7 \text{ cubes}}} \Rightarrow$ at least three of the same colour.

(b) Show that if 50 different integers (pigeons) are selected from the set {1, 2, 3, ..., 98}, there will be at least two integers whose sum is 99. (3 marks)

there are 49 'pigeonholes' that total 99
(1, 98), (2, 97), ... , (49, 50)
but 50 'pigeons'
therefore at least two in one of the 'pigeonholes' listed above.

TRINITY COLLEGE

Semester One Examination, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1
Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

SOLUTIONS

Student Number: In figures

In words ____________________

Your name ____________________

Time allowed for this section
Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

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Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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Section Two: Calculator-assumed **(106 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(6 marks)**

Three vectors are given by $\mathbf{a}=7\mathbf{i}$, $\mathbf{b}=6\mathbf{i}+9\mathbf{j}$ and $\mathbf{c}=x\mathbf{i}-5\mathbf{j}$.

(a) Use your calculator to determine the angle between $\mathbf{a}$ and $\mathbf{b}$, to the nearest degree. (2 marks)

Using CAS ClassPad

angle([7,0],[6,9]) ✓

$= \underline{\underline{56^\circ}}$ (nearest degree) ✓

OR

$\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{\binom{7}{0}\cdot\binom{6}{9}}{\sqrt{7^2}\sqrt{6^2+9^2}}\right)$

$\therefore \underline{\underline{\theta = 56^\circ}}$ (nearest degree)

OR since $\underset{\sim}{a}$ is horizontal, simply:

$\tan^{-1}\left(\frac{9}{6}\right) = \underline{\underline{56^\circ}}$ (nearest degree)

(b) Determine all possible values of x if $\mathbf{a}+\mathbf{c}$ and $\mathbf{b}+\mathbf{c}$ are perpendicular. (4 marks)

$(\underset{\sim}{a}+\underset{\sim}{c})\bullet(\underset{\sim}{b}+\underset{\sim}{c}) = 0$ ✓

$\Rightarrow \left(\binom{7}{0}+\binom{x}{-5}\right)\bullet\left(\binom{6}{9}+\binom{x}{-5}\right) = 0$

$\Rightarrow \binom{7+x}{-5}\bullet\binom{6+x}{4} = 0$ ✓

$\Rightarrow (x+7)(x+6) - 20 = 0$

$\Rightarrow x^2 + 13x + 22 = 0$ ✓

$\Rightarrow (x+11)(x+2) = 0$

$\therefore \underline{\underline{x=-11}}$ or $\underline{\underline{x=-2}}$ ✓

Could use: $m_1 m_2 = -1$

$\dfrac{-5}{x+7}\times\dfrac{4}{x+6} = -1$

$\Rightarrow -20 = -(x+7)(x+6)$

$\Rightarrow (x+7)(x-6)-20=0$

etc.

CAS ClassPad will deliver both results from this point.

N.B. CAS ClassPad: solve(dotP([x+7, x+6],[-5,4]) = 0, x)

only gives $\{x=-11\}$, yet $x=-2$ is also good. It's fun being smarter than a calculator!

Question 9 **(10 marks)**

(a) A multiple choice test has twelve questions and each question has three possible choices. If all questions are attempted, in how many ways can the test be answered? (2 marks)

$3^{12} = \underline{\underline{531441}}$

✓ ✓

(b) A set S contains all the integers between 3 and 102 inclusive. Determine

(i) how many numbers in set S are multiples of 7. (2 marks)

$\dfrac{102}{7} = 14.57$ (2d.p.) ✓

$\therefore \underline{\underline{14}}$ ✓

(ii) how many numbers in set S are multiples of 3 or 7. (3 marks)

$\dfrac{102}{3} = 34$ ✓ $\dfrac{102}{3\times7} = 4.86$ (2d.p.) ✓

Recall: $n(A\cup B) = n(A)+n(B) - n(A\cap B)$

$\therefore 34 + \underset{(i)}{14} - \underset{\text{Both}}{4}$

$= \underline{\underline{44}}$ ✓

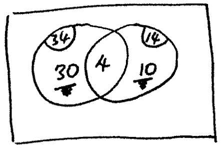
See Venn Diagram below

(iii) how many numbers in set S are multiples of either 3 or 7 but not both. (1 mark)

$34 + 14 - 2(4)$

$= 48 - 8$

$= \underline{\underline{40}}$ ✓ ie $30 + 10$

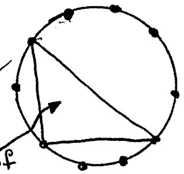


(c) Ten points are equally spaced around the circumference of a circle. Determine the number of simple (non-self-intersecting) convex polygons that can be formed by joining either three, four or five of these points with straight line segments. (2 marks)

$\binom{10}{3}+\binom{10}{4}+\binom{10}{5}$ ✓

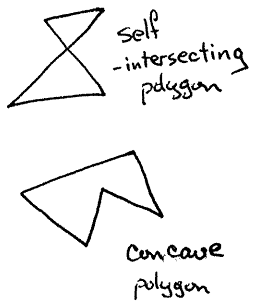
$= 120 + 210 + 252$

$= \underline{\underline{582}}$ ✓



One of the 120 triangles.

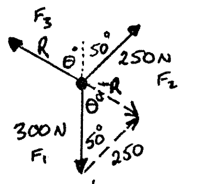
F.Y.I.:



Question 10 **(8 marks)**

Three forces are applied to a body. One has magnitude 300 N and acts due south. Another has magnitude 250 N and acts on a bearing of 050°.

(a) If all three forces are in equilibrium, determine the magnitude and direction of the third force. (4 marks)



For equilibrium

$-F_3 = F_1 + F_2$

$F_3 = -(F_1 + F_2)$

(see alternative method)

$R^2 = 300^2 + 250^2 - 2(300)(250)\cos 50^\circ$

$\Rightarrow R = \underline{236.8161}$ ✓

$\dfrac{\sin\theta^\circ}{250} = \dfrac{\sin 50^\circ}{236.8161}$

$\Rightarrow \quad \theta = \underline{53.97^\circ}$ ✓

$360^\circ - 54^\circ = 306^\circ$

An alternative approach:

$F_3 = -(F_1 + F_2)$

$= -\left(\begin{pmatrix} 0 \\ -300 \end{pmatrix} + \begin{pmatrix} 191.5 \\ 160.7 \end{pmatrix}\right)$

$= \begin{pmatrix} -191.5 \\ 139.3 \end{pmatrix}$

$|F_3| = \underline{237}\ N$

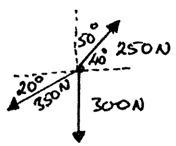
$\tan^{-1}\left(\frac{139.3}{-191.5}\right) = -36^\circ$

$270^\circ + 36^\circ$

$= \underline{306^\circ}$ (T)

$\therefore$ Magnitude is $\underline{237\,N}$ on bearing $\underline{306^\circ}$ ✓

(b) If the third force has a magnitude of 350 N and acts on a bearing of 250°, determine the magnitude and direction of the resultant force. (4 marks)



Beware: Bearing angle vs Standard position

Summing horizontal and vertical components:

$R = 300\begin{pmatrix} \cos 270^\circ \\ \sin 270^\circ \end{pmatrix} + 250\begin{pmatrix} \cos 40^\circ \\ \sin 40^\circ \end{pmatrix} + 350\begin{pmatrix} \cos 200^\circ \\ \sin 200^\circ \end{pmatrix}$ ✓

where all angles measured from standard position.

$\Rightarrow R = \begin{pmatrix} -137.3813 \\ -259.0101 \end{pmatrix}$

$\Rightarrow |R| = \underline{293.19}$

$\tan^{-1}\left(\dfrac{259.0101}{137.3813}\right) = 62.058$ ✓

$\therefore \quad 270 - 62.058$

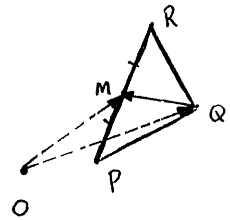
$= \underline{208^\circ}$ (nearest degree)

$\therefore$ Magnitude is $\underline{293\,N}$ on a $\underline{\text{bearing of } 208^\circ}$ ✓ ✓

Alternatively
Could use
Cosine Law
+ Sine Law
as in (a)

Question 11 **(6 marks)**

(a) A triangle PQR has vertices $P(1,\ 1)$, $Q(5,\ 3)$ and $R(3,\ 7)$. Determine the vector $\overrightarrow{QM}$, where M is the midpoint of side PR. (3 marks)



$\overrightarrow{QM} = \overrightarrow{OM} - \overrightarrow{OQ}$ ✓

$= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PR} - \overrightarrow{OQ}$

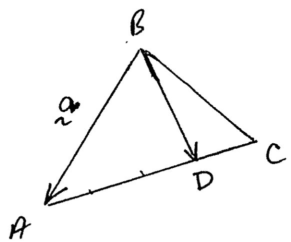
$= \overrightarrow{OP} + \frac{1}{2}(\overrightarrow{OR} - \overrightarrow{OP}) - \overrightarrow{OQ}$

$= \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OR} - \overrightarrow{OQ}$ ✓

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ ✓

(b) ABC is a triangle with point D on side AC such that $AD = \frac{3}{4}AC$. If $\overrightarrow{BA} = \mathbf{a}$ and $\overrightarrow{BD} = \mathbf{d}$, show that $\overrightarrow{BC} = \frac{1}{3}(4\mathbf{d} - \mathbf{a})$. (3 marks)



$\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$

$= \overrightarrow{BD} + \frac{1}{3}\overrightarrow{AD}$ ✓

$= \underset{\sim}{d} + \frac{1}{3}(\underset{\sim}{d} - \underset{\sim}{a})$

$= \underset{\sim}{d} + \frac{1}{3}\underset{\sim}{d} - \frac{1}{3}\underset{\sim}{a}$ ✓

$= \frac{4}{3}\underset{\sim}{d} - \frac{1}{3}\underset{\sim}{a}$

$= \frac{1}{3}(4\underset{\sim}{d} - \underset{\sim}{a})$ ✓

QED.

Question 12 **(8 marks)**

(a) Vectors $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude and vectors $\mathbf{a}$ and $\mathbf{c}$ are perpendicular, where $\mathbf{a}=\begin{bmatrix} m \\ n \end{bmatrix}$, $\mathbf{b}=\begin{bmatrix} -4 \\ 6 \end{bmatrix}$ and $\mathbf{c}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Determine the values of m and n. (3 marks)

Equal Magnitude

$m^2+n^2=(-4)^2+6^2$ and

Perpendicular

$2m+3n=0$ ✓

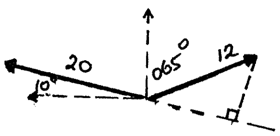
$\Rightarrow m^2+n^2=52$ ✓

Solve simult: $m=-6$ and $n=4$

or

$m=6$ and $n=-4$ ✓

(b) Determine the <u>scalar projection</u> of a velocity of 12 m/s on a bearing of 65° onto a velocity of 20 m/s on a <u>bearing</u> of 280°. (2 marks)



$12\cos(360^\circ-280^\circ+65^\circ)$

$=12\cos 145^\circ$ ✓

$=\underline{\underline{-9.83}}$ (2d.p.) ✓

The sketch suggests a negative result!

OR Use $12\cos 35^\circ$ and insert negative.

(c) The work done, in joules, by a force of $\mathbf{F}$ Newtons in changing the displacement of an object by $\mathbf{s}$ metres is given by the scalar product of $\mathbf{F}$ and $\mathbf{s}$.

A force acting on a bearing of 160° does work of 1 200 joules. If the object moved a distance of 350 cm on a bearing of 135°, determine the magnitude of the force. (3 marks)

3.5m

Given $|F|\,3.5\cos(160^\circ-135^\circ)=1200$ ✓

$\Rightarrow |F|=\dfrac{1200}{3.5\cos 25^\circ}$ ✓

$=\underline{\underline{378.3\text{ N}}}$ (1d.p.) ✓

Question 13 **(9 marks)**

(a) A triangle has vertices at $A(-3, 1)$, $B(-1, 4)$ and $C(5, 0)$.

(i) Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. (3 marks)

$\overrightarrow{AB}=\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$ ✓ $\overrightarrow{AC}=\begin{pmatrix}5\\0\end{pmatrix}-\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}8\\-1\end{pmatrix}$ ✓ $\overrightarrow{BC}=\begin{pmatrix}5\\0\end{pmatrix}-\begin{pmatrix}-1\\4\end{pmatrix}=\begin{pmatrix}6\\-4\end{pmatrix}$ ✓

(ii) Use a vector method to prove that triangle ABC is right-angled. (2 marks)

$\overrightarrow{AB}\cdot\overrightarrow{BC}=\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}6\\-4\end{pmatrix}$ ✓

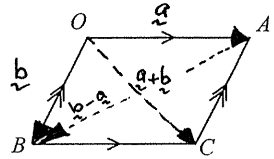
$=2\times6+3\times-4$

$=12-12$

$=0$ QED. ✓

Right Triangle @ Vertex B.

(b) Use a vector method to prove that if the diagonals of a parallelogram are perpendicular to each other, then the parallelogram is a rhombus. (4 marks)



See diagram for assigned vectors.

If $\overrightarrow{OC}\cdot\overrightarrow{AB}=0$

$\Rightarrow (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$ ✓

$\Rightarrow \cancel{\underset{\sim}{a}\cdot\underset{\sim}{b}}-\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}-\cancel{\underset{\sim}{b}\cdot\underset{\sim}{a}}=0$

$\Rightarrow \underset{\sim}{b}\cdot\underset{\sim}{b}=\underset{\sim}{a}\cdot\underset{\sim}{a}$ ✓

$\Rightarrow |\underset{\sim}{b}|^2=|\underset{\sim}{a}|^2$

$\Rightarrow |\underset{\sim}{b}|=|\underset{\sim}{a}|$ ✓

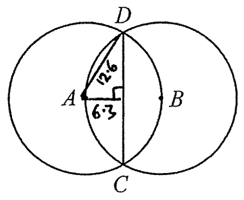
ie. equal magnitude (length)

$\therefore$ OABC is a rhombus

QED. ✓

Question 14 **(9 marks)**

(a) Two circles of radius 12.6 cm, with centres A and B as shown below, have a common chord CD. Determine, with justification, the length CD. (2 marks)

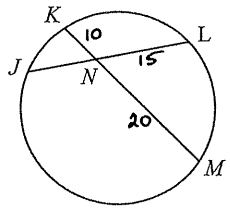


$\frac{1}{2}CD = \sqrt{12.6^2 - 6.3^2}$ ✓

$\Rightarrow CD = 2 \times 10.91$

$= \underline{\underline{21.82}}$ cm (2d.p.) ✓

(b) In the diagram below, $KN = 10$ cm, $LN = 15$ cm and $MN = 20$ cm. Determine, with justification, the exact length of JN. (4 marks)



Circle property of Chords.

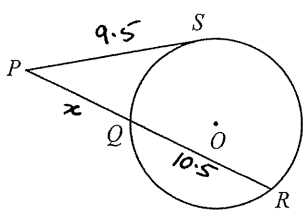
$15(JN) = 10 \times 20$ ✓✓

$\Rightarrow JN = \frac{200}{15}$

$\therefore JN = \underline{\underline{\frac{40}{3}}}$ cm $13\frac{1}{3}$ cm ✓✓

Assuming you have proved
$\triangle JKN \sim \triangle MLN$ (AAA)
$\Rightarrow \frac{JN}{20} = \frac{10}{15}$ which leads to the above Chord Property.

(c) Determine the length PQ, if the length of chord QR is 10.5 cm and the length of the tangent PS is 9.5 cm. (3 marks)



$PS^2 = PQ \times PR$ ✓

$\Rightarrow 9.5^2 = x\ (x + 10.5)$ ✓

$\Rightarrow x = -16.1$ or 5.6
Disgard.

$\therefore \underline{\underline{PQ = 5.6\text{cm}}}$ ✓

Question 15 **(10 marks)**

(a) A small body A has position (12, -3) m relative to another small body B. If a third small body C has position (-5, 6) relative to A, determine the position of B relative to C. (3 marks)

$_B\underset{\sim}{r}_C = \underset{\sim}{r}_B - \underset{\sim}{r}_C$ ✓

$= {}_B\underset{\sim}{r}_A - {}_C\underset{\sim}{r}_A$ treating A as the 'origin'.

$= -{}_A\underset{\sim}{r}_B - {}_C\underset{\sim}{r}_A$ ✓

$= -\begin{pmatrix}12\\-3\end{pmatrix} - \begin{pmatrix}-5\\6\end{pmatrix}$

$= \begin{pmatrix}-7\\-3\end{pmatrix}$ ✓

(b) To a cyclist moving with velocity (21, -5) km/h the wind appears to have velocity (-9, 3) km/h. Determine the true speed of the wind. (3 marks)

$_w\underset{\sim}{v}_c = \underset{\sim}{v}_w - \underset{\sim}{v}_c$ ✓

$\Rightarrow \begin{pmatrix}-9\\3\end{pmatrix} = \underset{\sim}{v}_w - \begin{pmatrix}21\\-5\end{pmatrix}$

$\Rightarrow \underset{\sim}{v}_w = \begin{pmatrix}-9\\3\end{pmatrix} + \begin{pmatrix}21\\-5\end{pmatrix}$

$= \begin{pmatrix}12\\-2\end{pmatrix}$ ✓

where $|\underset{\sim}{v}_w| = \sqrt{12^2 + (-2)^2}$

$= 2\sqrt{37}$

$= \underline{\underline{12.2}}$ km/h (1d.p) ✓

(c) A small ship is travelling with a constant speed of 14 knots on a bearing of 025° and another, larger ship is travelling with a constant speed of 17 knots on a bearing of 310°.

Determine the velocity of the large ship relative to the small ship. (4 marks)

ie Vector OR speed (knots) and bearing.

$$_L\underset{\sim}{V}_S = \underset{\sim}{V}_L - \underset{\sim}{V}_S \checkmark$$

$$= \begin{pmatrix} 17\cos 140^\circ \\ 17 \sin 140^\circ \end{pmatrix} - \begin{pmatrix} 14 \cos 65^\circ \\ 14 \sin 65^\circ \end{pmatrix} \checkmark\checkmark$$

$$= \begin{pmatrix} 17\cos 140^\circ - 14\cos 65^\circ \\ 17\sin 140^\circ - 14 \sin 65^\circ \end{pmatrix}$$

$$= \begin{pmatrix} -18.94 \\ -1.76 \end{pmatrix} \checkmark \text{ (2d.p.)}$$

It might be nice to leave the result in the format given ie. speed + bearing.

$$|_L\underset{\sim}{V}_S| = \sqrt{(-18.94)^2 + (-1.76)^2}$$

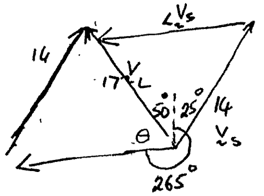
$$= 19 \text{ knots. (nearest knot)}$$

on bearing 265° (T)

since $\tan^{-1}\left(\frac{-1.76}{-18.94}\right) = 5^\circ$ (nearest degree)

ie. $(270-5)^\circ$

Alternatively



$$_L\underset{\sim}{V}_S = 14^2 + 17^2 - 2(14)(17)\cos 75^\circ$$

$$= 19 \text{ Knots} \quad (\checkmark\checkmark)$$

$$\frac{\sin\theta}{14} = \frac{\sin 75^\circ}{19.02}$$

$$\therefore \theta = 45.3$$

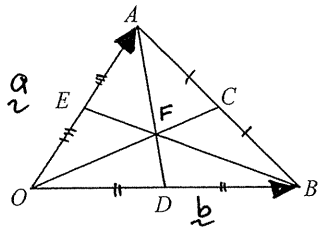
$\therefore$ Bearing 265° (nearest°) $(\checkmark\checkmark)$

Question 16 **(9 marks)**

The medians of triangle OAB are OC, AD and BE, as shown below.

(A median joins a vertex to the midpoint of the opposite side of the triangle).

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



(a) Prove that $\overrightarrow{OC} + \overrightarrow{AD} + \overrightarrow{BE} = \mathbf{0}$. (4 marks)

Take LHS.

$$= \overrightarrow{OC} + \overrightarrow{AD} + \overrightarrow{BE}$$

$$= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB} + \overrightarrow{AO} + \tfrac{1}{2}\overrightarrow{OB} + \overrightarrow{BO} + \tfrac{1}{2}\overrightarrow{OA} \checkmark$$

$$= \underset{\sim}{a} + \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a}) - \underset{\sim}{a} + \tfrac{1}{2}\underset{\sim}{b} - \underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{a} \checkmark\checkmark$$

$$= \underset{\sim}{a} + \tfrac{1}{2}\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{a} - \underset{\sim}{a} + \tfrac{1}{2}\underset{\sim}{b} - \underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{a}$$

$$= 0$$

$$= \text{RHS.} \quad \text{QED.} \checkmark$$

(b) The centroid, F, is the point of intersection of the medians.

Determine $\overrightarrow{AF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (5 marks)

(*Hint: Let $\overrightarrow{EF} = h\overrightarrow{EB}$, $\overrightarrow{OF} = k\overrightarrow{OC}$ and first solve for h and k*)

Using hint:

$\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF} = \overrightarrow{AO} + \overrightarrow{OF}$ ✓

$\Rightarrow -\frac{1}{2}\underset{\sim}{a} + h\overrightarrow{EB} = -\underset{\sim}{a} + k\overrightarrow{OC}$

$\Rightarrow -\frac{1}{2}\underset{\sim}{a} + h\left(\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}\right) = -\underset{\sim}{a} + k\left(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right)$

$\Rightarrow -\frac{1}{2}\underset{\sim}{a} + h\underset{\sim}{b} - \frac{1}{2}h\underset{\sim}{a} = -\underset{\sim}{a} + \frac{1}{2}k\underset{\sim}{a} + \frac{1}{2}k\underset{\sim}{b}$ ✓

$\Rightarrow \left(-\frac{1}{2} - \frac{1}{2}h\right)\underset{\sim}{a} + h\underset{\sim}{b} = \left(-1 + \frac{1}{2}k\right)\underset{\sim}{a} + \frac{1}{2}k\underset{\sim}{b}$

Equating scalars and solving simultaneously:

$-\frac{1}{2} - \frac{1}{2}h = -1 + \frac{1}{2}k$ and $h = \frac{1}{2}k$ ✓

$\Rightarrow -1 - h = -2 + k$

$\Rightarrow h + k = 1$

$\therefore h = \frac{1}{3}, \quad k = \frac{2}{3}$ ✓

Possible to state these given centroid divides median in ratio 2:1 from vertex.

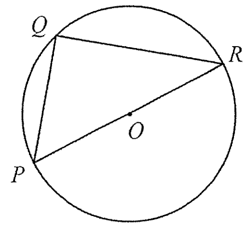
$\therefore \overrightarrow{AF} = -\frac{1}{2}a + \frac{1}{3}\left(\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}\right)$

$= -\frac{1}{2}a + \frac{1}{3}\underset{\sim}{b} - \frac{1}{6}\underset{\sim}{a}$

$= \frac{1}{3}\underset{\sim}{b} - \frac{2}{3}\underset{\sim}{a}$ ✓ as required.

Question 17 **(8 marks)**

(a) The diagram shows a triangle with vertices P, Q and R that lie on a circle with centre O. Chord PR passes through O. Prove, by contradiction, that angle QPR is acute. (4 marks)



Assume $\angle QPR$ is not acute ie. $\angle QPR \geq 90^\circ$ ✓

$\angle PQR = 90^\circ$ ✓ angle in a semicircle.

$\angle QRP > 0^\circ$ as it is the third angle of a triange

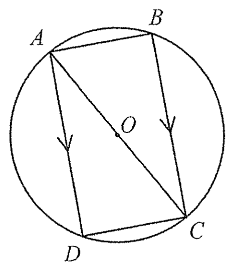
$\angle QPR + \angle PQR + \angle QRP > 180^\circ$ ✓

ie. $\geq 90^\circ \quad 90^\circ \quad > 0^\circ$

This contradicts the angles of a triangle add to 180° ✓

$\therefore$ Assumption must be false and $\angle QPR < 90^\circ$ ie. acute QED.

(b) In the diagram below, O is the centre of the circle on which points A, B, C and D lie. Chord AC passes through O and BC is parallel to AD. Prove that the quadrilateral $ABCD$ is a rectangle. (4 marks)



$\angle ABC = \angle CDA = 90°$ both angles in a semicircle. ✓ (A)

$\angle BCA = \angle DAC$ alternate angles ✓ (A)

AC common to $\triangle ABC$ and $\triangle CDA$ (S)

Hence $\triangle ABC$ and $\triangle CDA$ are <u>congruent</u> ✓ (ASA)

Hence $AB = CD$ and $BC = DA$

$\angle BAD = \angle DCB = 90°$ Cointerior with $\angle ABC$ and $\angle CDA$

Thus $ABCD$ is a <u>rectangle</u>. ✓

QED.

Question 18 **(8 marks)**

(a) A small coach has 24 seats, arranged in six rows of four seats each, with two seats in each row on either side of the central aisle. A group of passengers consisting of ten males and nine females board the bus.

(i) Determine how many combinations of empty seats are possible once everyone has sat down. (1 mark)

$$\binom{24}{5} = 42504 \checkmark$$

(ii) How many <u>fewer</u> combinations are there if the females all sit on one side of the aisle and the males all sit on the other side? (3 marks)

$$42504 - \binom{12}{2}\binom{12}{3} \checkmark$$

$$= 42504 - 14520 \checkmark$$

$$= \underline{\underline{27984}} \text{ fewer seats.} \checkmark$$

(b) Determine the number of possible four letter permutations of the letters of the word

(i) RELOAD. (1 mark)

$$^6P_4 = 360 \checkmark$$

(ii) RELOADED. Select and arrange (divide out repeats) (3 marks)

All different + 2 E's + 2 D's + 2 E's and 2 D's

$$\underset{(i)}{360} + \underbrace{\binom{2}{2}\binom{5}{2}\cdot\frac{4!}{2!}}_{\text{select and arrange}} + \binom{2}{2}\binom{5}{2}\frac{4!}{2!} + \binom{2}{2}\binom{2}{2}\frac{4!}{2!2!} \checkmark$$

E others, D others, E D

$$= 360 + 120 + 120 + 6 \checkmark$$

$$= \underline{\underline{606}} \checkmark$$

Question 19 **(7 marks)**

A small boat has to travel across a river from A to B, where $OA = 60\mathbf{i} + 35\mathbf{j}$ m and $OB = 356\mathbf{i} - 125\mathbf{j}$ m. A uniform current of $-1.5\mathbf{i} + 2.5\mathbf{j}$ m/s is flowing in the river and the boat can maintain a steady speed of 4 m/s.

(a) Determine, in the form $a\mathbf{i} + b\mathbf{j}$, the velocity vector the small boat should set to travel directly from A to B. (5 marks)

$$\vec{AB} = \begin{pmatrix} 356 \\ -125 \end{pmatrix} - \begin{pmatrix} 60 \\ 35 \end{pmatrix}$$

$$= \begin{pmatrix} 296 \\ -160 \end{pmatrix} \checkmark \quad \text{Let } \underset{\sim}{v}_{Boat} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Now $t\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1.5 \\ 2.5 \end{pmatrix}\right) = \begin{pmatrix} 296 \\ -160 \end{pmatrix}$ ✓ and $\underline{x^2 + y^2 = 4^2}$ ✓

$\Rightarrow t\begin{pmatrix} x-1.5 \\ y+2.5 \end{pmatrix} = \begin{pmatrix} 296 \\ -160 \end{pmatrix}$

$\Rightarrow t = \dfrac{296}{x-1.5}$ and $t = \dfrac{-160}{y+2.5}$ Solve simultaneously using CAS ClassPad.

$\Rightarrow \underline{296(y+2.5) = -160(x-1.5)}$ ✓

$x = -3.9736$ or $x = 2.5604$

$y = 0.4587$ $y = -3.0732$ as $t \geq 0$

Disregard as $t < 0$

$\therefore \underset{\sim}{v} = \underline{2.56\,\underset{\sim}{i} - 3.07\,\underset{\sim}{j}}$ ✓

(b) Calculate how long the journey will take. (2 marks)

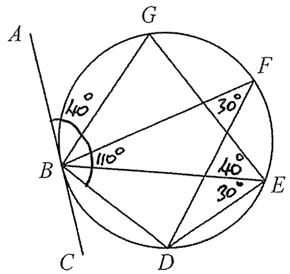
$(2.5604 - 1.5)t = 296$ ✓

$\Rightarrow t = \underline{279.1}$ sec. ✓

4 min 39 sec (nearest)

Question 20 **(8 marks)**

(a) In the diagram below, AC is a tangent to the circle at B. If $\angle ABG = 40°$, $\angle GBF = 25°$ and $\angle BFD = 30°$, determine the size of angle DBF. (4 marks)



$\angle BEG = \angle ABG = 40°$ (Alt. segment) ✓

$\angle BED = \angle BFD = 30°$ (Standing on same chord) ✓

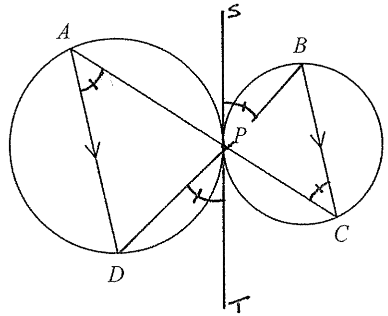
$\angle GBD = 110°$ (cyclic quad.) ✓

$\angle DBF = 110° - \angle GBF$

$= 110° - 25°$

$= \underline{85°}$ ✓

(b) In the diagram below, the line AC passes through the point P, where both circles touch each other. The line AD is parallel to line BC. Prove that the points B, P and D are collinear. (4 marks)



Add tangent ST at P. ✓

$\angle PCB = \angle BPS$ Alt. segment.

$\angle PAD = \angle DPT$ " " ✓

$\angle PCB = \angle PAD$ Alt. angles. given parallel lines.

$\therefore \angle BPS = \angle DPT$ ✓

Hence BD is a straight line making vertically opposite angles with AC and so

B, P and D are collinear ✓

QED.

End of questions